Electromagnetic waves produced by a travelling current pulse with high-frequency filling

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Abstract
Solving the inhomogeneous Maxwell equations in the spacetime domain, the formation of electromagnetic waves by current pulses with high-frequency filling of finite duration moving along a straight segment with luminal or subluminal speed is investigated. Being akin to a well-known description of the travelling-wave antenna, this model is capable of rough description of artificial and natural line radiators of different nature, including radiation accompanying absorption of high-energy photons or particles in a medium. For arbitrary slowly varying envelope of the current pulse, a closed-form quadrature expression is obtained for the magnetic component of the field, which enables the entire field to be reconstructed at long distances from the source. This expression describes well peculiarities of non-stationary electromagnetic waves due to source-current filling: directionality, frequency transform and beating. Results obtained may be readily used in the case of scalar waves.

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1. Introduction

In the present work, we extend the well-known model of the travelling-wave line source [1, 2] to the case of the formation of electromagnetic waves by a pulse with high-frequency filling moving along a line segment whose physical realization depends on physical realization of the model. The velocity of the high-frequency (carrier) wave differs from the velocity of the front and the trailing edge of the pulse. Generation of the electromagnetic waves is described by solving the inhomogeneous Maxwell equations directly in the spacetime domain: first, the electric and magnetic field vectors are expressed in terms of one scalar function and then the resulting hyperbolic-type PDE is solved using incomplete separation of variables and the Riemann formula (see [3] for details).

Investigation of a pulse with high-frequency filling moving along a line segment was stimulated by the problem of launching directional scalar and electromagnetic waves (missiles) as well as by the results of experimental investigation of superradiation waveforms reported in [4]. One of the possible realizations of such pulses is linked with the absorption of a spike pulse of hard radiation by a medium [5] producing anisotropic distribution of photoelectrons with maximum in the direction of photon pulse propagation. Coordinated motion of the electrons gives rise to a macroscopic source current whose distribution patterns propagate with a luminal or superluminal speed [6, 7]. Consideration of finite-mass particles instead of high-energy photons results in the same model with the subluminal current propagation. As is discussed in [8], the high-frequency filling that provides coherence to the emanated radiation may result from stimulated relaxation of a medium preliminary set in a nonequilibrium state.

The model in question can roughly describe a number of traditional artificial as well as natural line radiators and, being characterized by two different velocities—the phase velocity of the carrier wave and the source-pulse velocity—explains characteristic features observed in laboratory and natural conditions for waves emanated by sources with filling: their directionality, frequency transform and beating. Analytical results obtained for some important particular cases enable us to describe the formation of interesting localized wave structures.

2. Basic relations

We assume that the line of current propagation coincides with the z-axis of a cylindrical coordinate system ρ, φ, z whose origin is located in the starting point of the current pulse. Since the problem has the axial symmetry, the current density vector $\mathbf{j}$ can be described as

$$\mathbf{j} = \mathbf{e}_z j_z, \qquad j_z = \frac{\delta(\rho)}{2\pi\rho} h(z) h(l - z) I(z, \tau), \qquad \tau > 0 \quad \text{and} \quad j_z \equiv 0, \tau < 0 \qquad (1)$$

where $\tau = ct$ is the time variable, c is the velocity of light, $\delta(\rho)$ and

$$h(z) = \begin{cases} 1, & z > 0 \\ 0, & z < 0 \end{cases}$$

are the Dirac distribution and the Heaviside step function, respectively, and l is the radiator length.

We will take the total current distribution in the form

$$I_\mp = h(\beta\tau - z) h(z - \beta\tau + \beta T) U(z, \tau) \exp[ik(\tau \mp z)]. \qquad (2)$$

Here, the parameter T is the current-pulse duration, $\beta = v/c \in (0, 1]$, where v is the velocity of the pulse, $U(z, \tau)$ is an arbitrary differentiable function and $k = \omega/c$, where ω is the angular frequency. The modulation term is represented in a standard complex exponent notation; from here on, the real part of the complex expressions should be attributed to the co-sinusoidal modulation and the imaginary to the sinusoidal modulation.

Expressing the components of the electric induction and the magnetic field strength vectors, $\mathbf{D}$ and $\mathbf{H}$, via the Whittaker–Bromwich potential u [9, 10]

$$D_\rho = \frac{\partial^2}{\partial\rho\partial z} u, \qquad D_z = -\frac{\partial^2}{\partial\tau^2} u + \frac{\partial^2}{\partial z^2} u, \qquad H_\varphi = -c\frac{\partial^2}{\partial\rho\partial\tau} u \qquad (3)$$

and assuming that the initial conditions are

$$\mathbf{D} = \mathbf{H} = 0, \qquad \tau < 0 \qquad (4)$$

one can obtain from the Maxwell equations the following initial-value problem:

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) - \frac{\partial^2}{\partial z^2}\right)\Psi = \frac{1}{c}j_z, \qquad \Psi \equiv 0, \tau < 0 \tag{5}$$

where the scalar wavefunction $\Psi = \partial u/\partial \tau$ represents the temporal derivative of the Whittaker–Bromwich potential. The general solution of the above problem was constructed by Borisov and Utkin [11] using the Smirnov method of incomplete separation of variables [12]. The variable ρ is separated via the Fourier–Bessel transform ($\rho \to s$) and the solution of the resulting problem is constructed as a two-dimensional integral with the help of the Riemann formula [13] (the Riemann function of the PDE, $R(\tau, z; \tau', z') = J_0(s\sqrt{(\tau - \tau')^2 - (z - z')^2})$, is known). Due to the localized nature of the source term and the specific properties of the Bessel functions composing the integrand, the final expression for the wavefunction in the ρ, z, τ domain reduces to a one-dimensional integral

$$\Psi_{\mp} = \frac{1}{4\pi c}e^{ik\tau}\int_{\Phi_2}^{\Phi_1}dz'\frac{1}{[\rho^2 + (z - z')^2]^{1/2}}U(z', \tau - [\rho^2 \pm (z - z')^2]^{1/2})$$
$$\times \exp(-ik\{[\rho^2 + (z - z')^2]^{1/2} \pm z'\}). \tag{6}$$

Hereinafter, if necessary, the wavefunctions and the components of the electromagnetic field vectors produced by the total currents I_- and I_+ are denoted by the subscripts $(-)$ and $(+)$.

The parameters l, T, β, the observation time τ and location ρ, z determine the limits of integration $\Phi_{1,2}$. The following different cases can be distinguished [11]:

(i) $0 < \tau - r < T$ and $\tau - r_l < \dfrac{l}{\beta}$

(ii) $\tau - r > T$ and $\tau - r_l < \dfrac{l}{\beta}$

(iii) $0 < \tau - r < T$ and $\tau - r_l > \dfrac{l}{\beta}$

(vi) $r < \tau - T < \dfrac{l}{\beta} + r_l$ and $\tau - r_l > \dfrac{l}{\beta}$. $\tag{7}$

Here, $r = (\rho^2 + z^2)^{1/2}$ and $r_l = [\rho^2 + (z - l)^2]^{1/2}$; for $\tau - r < 0$ and $\tau - T > l/\beta + r_l$, we have $\Psi \equiv 0$. Note that one can obtain all the above inequalities by the geometric treatment using the causality principle. They correspond to interrelations between

- the positions of two spherical wavefronts emanated from the radiator's beginning ($\rho = 0, z = 0$) by the front and the back of the source pulse at the instants $\tau = 0$ and $\tau = T$, respectively;
- the positions of two spherical wavefronts emanated from the radiator's end ($\rho = 0, z = l$) by the front and the back of the source pulse at the moments $\tau = l/\beta$ and $\tau = l/\beta + T$, respectively;
- the distance from the radiator's beginning to the observation point r and the distance from the radiator's end to the observation point r_l.

Initial inequalities are given in table 1.

The expressions for the functions $\Phi_{1,2}$ are given in [11]. Using simple algebraic analysis, one can check that there exists no combination of the pulse parameters that realizes all four cases (i)–(iv): with the variation of the time parameter τ from zero to infinity, the order of application of the cases is reduced to (i), (ii) and (iv) ($T + r < l/\beta + r_l$, a *short* pulse) or (i), (iii) and (iv) (the opposite case, a *long* pulse)—see section 4 of [11] for detailed discussion.

Table 1. Initial inequalities that define the integration limits for solution (6).

Case	Wavefront 1	Wavefront 2	Wavefront 3	Wavefront 4
$\psi \equiv 0$	$\tau < r$	$\tau - T < r$	$\tau - l/\beta < r_l$	$\tau - T - l/\beta < r_l$
(i)	$\tau > r$	$\tau - T < r$	$\tau - l/\beta < r_l$	$\tau - T - l/\beta < r_l$
(ii)	$\tau > r$	$\tau - T > r$	$\tau - l/\beta < r_l$	$\tau - T - l/\beta < r_l$
(iii)	$\tau > r$	$\tau - T < r$	$\tau - l/\beta > r_l$	$\tau - T - l/\beta < r_l$
(iv)	$\tau > r$	$\tau - T > r$	$\tau - l/\beta > r_l$	$\tau - T - l/\beta < r_l$
$\psi \equiv 0$	$\tau > r$	$\tau - T > r$	$\tau - l/\beta > r_l$	$\tau - T - l/\beta > r_l$

Relations (3) and (6) determine immediately the magnetic field strength component while definition of the electric induction components in the near zone requires calculation of the Whittaker–Bromwich potential itself, which leads to integration with respect to the time variable. This calculation requires separate consideration that is specific to physical realization of the model (due to the initial conditions for which the charge distribution must be specified) and will not be discussed in the scope of the present work. The electric induction components (and, consequently, the entire electromagnetic field) in the far zone, $r \gg l$, can be found from the known magnetic field components (see, for example, [14]). All the subsequent results can easily be extended to the case of scalar waves, in which the wave process is completely described by the function Ψ.

3. The magnetic field strength produced by the total current distribution I_-

The continuous factor of the total current distribution I_- has the form

$$U(z, \tau) \exp[ik(\tau - z)]. \tag{8}$$

One can write the function $U(z, \tau)$ as a function of the variables $\tau - z$ and $\tau + z$. Then, the factor $U[z, \tau - [\rho^2 + (z - z')^2]^{1/2}]$ in integral (6) turns into $U[\tau - [\rho^2 + (z - z')^2]^{1/2} - z', \tau - [\rho^2 + (z - z')^2]^{1/2} + z']$. Using the change of the integration variable $\xi_1' = \tau - [\rho^2 + (z - z')^2]^{1/2} - z'$ we obtain

$$\Psi = \frac{1}{4\pi c} \int_{\Phi_1}^{\Phi_2} \mathrm{d}\xi_1' \frac{1}{\tau - z - \xi_1'} U\left(\xi_1', \tau + z - \frac{\rho^2}{\tau - z - \xi_1'}\right) e^{ik\xi_1'}. \tag{9}$$

Assuming that the function U is a slowly varying function, the expression

$$q(\xi_1') = \frac{1}{\tau - z - \xi_1'} U\left(\xi_1', \tau + z - \frac{\rho^2}{\tau - z - \xi_1'}\right)$$

is the continuous function and $q'(\xi_1')$ is the absolutely integrable function, one can obtain from (9) the description of the wavefunction by integration by parts [15]

$$\Psi_- \cong \frac{i}{4\pi ck} \left[\frac{1}{\tau - z - \Phi_1} U\left(\Phi_1, \tau + z - \frac{\rho^2}{\tau - z - \Phi_1}\right) e^{ik\Phi_1} \right.$$
$$\left. - \frac{1}{\tau - z - \Phi_2} U\left(\Phi_2, \tau + z - \frac{\rho^2}{\tau - z - \Phi_2}\right) e^{ik\Phi_2} \right] + o\left(\frac{1}{kl}\right). \tag{10}$$

Hence, using the relation $H_\varphi = -c\partial\Psi/\partial\rho$ (3) we have

$$H_\varphi \cong \frac{1}{4\pi} \left[\frac{1}{\tau - z - \Phi_1} \frac{\partial\Phi_1}{\partial\rho} U\left(\Phi_1, \tau - z - \frac{\rho^2}{\tau - z - \Phi_1}\right) e^{ik\Phi_1} \right.$$
$$\left. - \frac{1}{\tau - z - \Phi_2} \frac{\partial\Phi_2}{\partial\rho} U\left(\Phi_2, \tau + z - \frac{\rho^2}{\tau - z - \Phi_2}\right) e^{ik\Phi_2} \right]. \tag{11}$$

In the particular case when the function U in relation (8) is the function of only one variable $\tau - z$, the above result becomes an exact solution of the electrodynamic problem.

Case (i). In this case, we have

$$\Phi_1 = \Phi_1^{(i)} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}\left[\tau_\beta - \sqrt{\rho^2 + z_\beta^2}\right], \qquad \Phi_2 = \Phi_2^{(i)} = \tau - \sqrt{\rho^2 + z^2} \qquad (12)$$

where τ_β and z_β are the variables of the frame of reference moving with the velocity $v = \beta c$ in the direction of the current-pulse motion

$$\tau_\beta = \frac{\tau - \beta z}{\sqrt{1-\beta^2}}, \qquad z_\beta = \frac{z - \beta\tau}{\sqrt{1-\beta^2}}.$$

From (11) and (12), we get

$$H_\varphi \cong \frac{1}{4\pi}\frac{\rho}{r(r-z)}U(0)\exp[ik(\tau - r)] - \frac{1}{4\pi}\frac{\rho}{r_\beta(r_\beta - z_\beta)}U(\beta)$$

$$\times \exp\left[ik\left(\frac{1-\beta}{1+\beta}\right)^{1/2}(\tau_\beta - r_\beta)\right] = H_{\varphi-}(0) - H_{\varphi-}(\beta) \qquad (13)$$

where $r = \sqrt{\rho^2 + z^2}$, $r_\beta = \sqrt{\rho^2 + z_\beta^2}$, $U(0) = U(\tau - r, \tau - r)$ and

$$U(\beta) = U\left(\left(\frac{1-\beta}{1+\beta}\right)^{1/2}(\tau_\beta - r_\beta), \left(\frac{1+\beta}{1-\beta}\right)^{1/2}(\tau_\beta - r_\beta)\right).$$

Using the spherical coordinates r, θ, ϕ and $r_\beta, \theta_\beta, \phi$, defined so that $\rho = r\sin\theta = r_\beta\sin\theta_\beta$, $z = r\cos\theta$ and $z_\beta = r_\beta\cos\theta_\beta$, one can represent the factors $\rho/(r-z)$ and $\rho/(r_\beta - z_\beta)$ as the angular coefficients $R(\theta) = \sin\theta/(1-\cos\theta)$ and $R(\theta_\beta) = \sin\theta_\beta/(1-\cos\theta_\beta)$. Note that r_β and θ_β are the time-dependent functions. The angular coefficients in the above relation are convenient for the preliminary analysis of the spacetime structure of generated waves. However, the detailed investigation requires the transition to the variables ρ, φ, z of the initial frame of reference.

One can see from expressions (13) that

1. The angular factors $R(\theta)$ and $R(\theta_\beta)$ tend to zero as $\theta \to \pi$ and $\theta_\beta \to \pi$, respectively, hence $H_\varphi = 0$, if $z \in (0, -\infty)$, $\rho = 0$.
2. The factors $R(\theta)$ and $R(\theta_\beta)$ are infinite when $\theta \to 0$ and $\theta_\beta \to 0$,, and therefore it is necessary to analyse the limit of H_φ in detail. It is evident that H_φ is equal to zero, if $U = \text{const}$ and $k = 0$.
3. When $\theta = 0$ and $\theta_\beta = \pi$, the factors $R(\theta)$ and $R(\theta_\beta)$ become infinite and zero, respectively, hence the segment $z \in (0, \beta\tau)$, $\rho = 0$ is the domain where H_φ is infinite (the domain of the source).
4. It is possible to achieve directionality of the waves produced by the current pulse I_-, provided that the angles θ and θ_β are small.
5. Using the variables τ, ρ, z, one can check that the factor

$$\frac{1}{r_\beta}R(\theta_\beta) \cong \frac{(1-\beta^2)\rho}{2(\beta\tau - z)^2} + o[(1-\beta^2)^2]$$

and the argument

$$\left(\frac{1-\beta}{1+\beta}\right)^{1/2}(\tau_\beta - r_\beta) \cong \frac{1-\beta}{1+\beta}\left[\tau + z - \rho^2\frac{1+\beta}{2(\beta\tau - z)}\right] + o(1-\beta^2)$$

tend to zero as $\beta \to 1$ (here $z < \beta\tau$). Hence, the second term in expression (13), $H_{\varphi-}(\beta)$, equals zero, if the factor $U(\beta)$ remains finite as $\beta \to 1-$ (the superluminar velocities are not considered in this work and from here on we denote $\beta \to 1-$ as $\beta \to 1$).

6. The frequency transform is observed in the second term $H_{\varphi-}(\beta)$ of expression (13). Rewriting expression $k((1-\beta)/(1+\beta))^{1/2}(\tau_\beta - r_\beta)$ in the variables of the initial frame, one can find the limiting values of the transformed frequency: $\omega_\beta = \omega$ if $\theta = 0 (z > \beta\tau, \rho = 0)$ and $\omega_\beta = ((1-\beta)/(1+\beta))\omega$ when $\theta = \pi (z < 0, \rho = 0)$: the frequency shift towards the red spectrum region. Here, $\omega_\beta = ck_\beta$ is the frequency of the initial frame. Note that formula (13) yields the maximum value of the red shift for the radiation emitted in the direction opposite to the direction of the movement of the current-pulse front.

Case (ii). The limits of integration are

$$\Phi_1^{(ii)} = \Phi_1^{(i)}, \qquad \Phi_2^{(ii)} = T + \left(\frac{1-\beta}{1+\beta}\right)^{1/2} (\tau_T - r_T). \tag{14}$$

In the expression for the upper limit, we use the variable transform

$$\tau_T = \frac{\tau - T - \beta z}{\sqrt{1-\beta^2}}, \qquad z_T = \frac{z - \beta(\tau - T)}{\sqrt{1-\beta^2}} \quad \text{and define} \quad r_T = \sqrt{\rho^2 + z_T^2}. \tag{15}$$

From relations (11), (14) and (15) we can write

$$H_\varphi \cong \frac{1}{4\pi} \frac{\rho}{r_T(r_T - z_T)} U(T) \exp\left\{ik\left[T + \left(\frac{1-\beta}{1+\beta}\right)^{1/2} (\tau_T - r_T)\right]\right\} - H_{\varphi-}(\beta)$$

$$= H_{\varphi-}(T) - H_{\varphi-}(\beta)$$

$$U(T) = U\left[T + \left(\frac{1-\beta}{1+\beta}\right)^{1/2} (\tau_T - r_T), T + \left(\frac{1+\beta}{1-\beta}\right)^{1/2} (\tau_T - r_T)\right]. \tag{16}$$

Using the spherical coordinates r_T, θ_T, φ defined so that $\rho = r_T \sin\theta_T$ and $z_T = r_T \cos\theta_T$, one can represent the factor $\rho/(r_T - z_T)$ as an angular coefficient

$$R(\theta_T) = \frac{\sin\theta_T}{1 - \cos\theta_T}.$$

The results of the preliminary analysis are the following:

1. In relation (16), the angular factors $R(\theta_\beta)$ and $R(\theta_T)$ are equal to zero if $\theta_\beta = \theta_T = \pi$, hence $H_\varphi = 0$ on the line $\rho = 0$ when $z < \beta(\tau - T)$.
2. When $\theta_\beta = \pi$ while $\theta_T = 0$, the magnetic field component is infinite. It is not surprising bearing in mind that the segment $[\beta(\tau - T), \beta\tau]$ of the $0z$-axis is the source domain.
3. The wave directionality is observed when θ_β and θ_T are small but not equal to zero.
4. We have the same frequency transform for the both terms, the limiting values of the frequencies $\omega_\beta = \omega((1-\beta)/(1+\beta))$ and $\omega_\beta = \omega$ are observed for $\theta = \pi$ and $\theta = 0$ (the 'red' shift of the frequency).
5. The terms $H_{\varphi-}(\beta)$ and $H_{\varphi-}(T)$ tend to zero as $\beta \to 1$, hence H_φ is equal to zero.

Case (iii). The expressions for the limits of integration are

$$\Phi_1^{(iii)} = \tau - l - \sqrt{\rho^2 + (z - l)^2}, \qquad \Phi_2^{(iii)} = \Phi_2^{(i)}. \tag{17}$$

Using the above relations as well as the expressions (3) and (11), one can obtain

$$H_\varphi \cong H_{\varphi-}(0) - \frac{1}{4\pi} \frac{\rho}{r_l(r_l - z_l)} U(l) \exp[ik(\tau - r_l - l)] = H_{\varphi-}(0) - H_{\varphi-}(l) \tag{18}$$

where the factor $U(l) = U(\tau - r_l - l, \tau - r_l + l)$. Here, we use notation $z_l = z - l$ and $r_l = \sqrt{\rho^2 + z_l^2}$. Again, in the spherical coordinate system r_l, θ_l, φ, in which $\rho = r_l \sin\theta_l$ and $z_l = r_l \cos\theta_l$, the angular factor can be expressed as

$$R(\theta_l) = \frac{\rho}{r_l - z_l} = \frac{\sin\theta_l}{1 - \cos\theta_l}.$$

We see that H_φ is infinite on the segment $z \in [0, l]$ of the line $\rho = 0$ (the source domain) and the frequency transform is not observed.

Case (iv). In this case, we have the limits of integration

$$\Phi_1^{(iv)} = \Phi_1^{(iii)}, \qquad \Phi_2^{(iv)} = \Phi_2^{(ii)}. \tag{19}$$

Using relation (11) as well as the terms $H_{\varphi-}(T)$ and $H_{\varphi-}(l)$ in expressions (16) and (18), we can write the magnetic field strength component as

$$H_\varphi \cong H_{\varphi-}(T) - H_{\varphi-}(l). \tag{20}$$

In the spacetime domain (iv), the above solution is infinite on the segment $z \in [\beta(\tau - T), l]$ of the line $\rho = 0$. In the first term of the above relation we have the frequency transform, but this term is zero if $\beta = 1$.

We obtain expressions (13), (16), (18) and (20) from relation (11), hence the above results are the exact solutions of the Maxwell equations in the particular case of the factor U in (8) expressed in the form $U = U(\tau - z)$. Otherwise, the approximate relations describing the terms $H_{\varphi-}(\beta)$ and $H_{\varphi-}(T)$ in the limit $\beta \to 1$ require special consideration for each slowly varying function $U(\tau, z) \neq U(\tau - z)$.

Emanated wave structure for the case of a rectangular envelope $U(z, \tau) = U_0 = \text{const}$ is depicted in figure 1. As far as conditions (7) and even the condition defining the pulse type are inherently local, this figure illustrates all the structural forms of the magnetic field—cases (i)–(iv), equations (13), (16), (18) and (20)—for both sort and long pulse types: different formulae are applied for different spacetime domains. The chosen parameters of wave generation, $kl = 200$, $T/l = 0.3$ and $\beta = 0.9$, are close to those of the experiments described by Egorov *et al* [4]. Phenomena linked with the wave localization and frequency transform will be discussed in details in section 5.

4. The magnetic field strength produced by the total current distribution I_+

Here, we assume that the continuous factor of the current distribution (2) is

$$U(\tau, z) \exp[ik(\tau + z)]. \tag{21}$$

Changing the integration variable in expression (9) to $x = \rho^2/(\tau - z - \xi_1')$ we obtain

$$\Psi_+ = \frac{1}{4\pi c} \exp(ik(\tau + z)) \int_{F_1}^{F_2} dx \frac{1}{x} U\left(\frac{\tau - z - \rho^2}{x}, \tau + z - x\right) e^{-ikx} \tag{22}$$

where the integration limits are

Case (i):

$$F_1^{(i)} = \left(\frac{1 + \beta}{1 - \beta}\right)^{1/2} (r_\beta + z_\beta) \qquad \text{and} \qquad F_2^{(i)} = r + z. \tag{23}$$

Case (ii):

$$F_1^{(ii)} = F_1^{(i)} \qquad \text{and} \qquad F_2^{(ii)} = \left(\frac{1 + \beta}{1 - \beta}\right)^{1/2} (r_T + z_T). \tag{24}$$
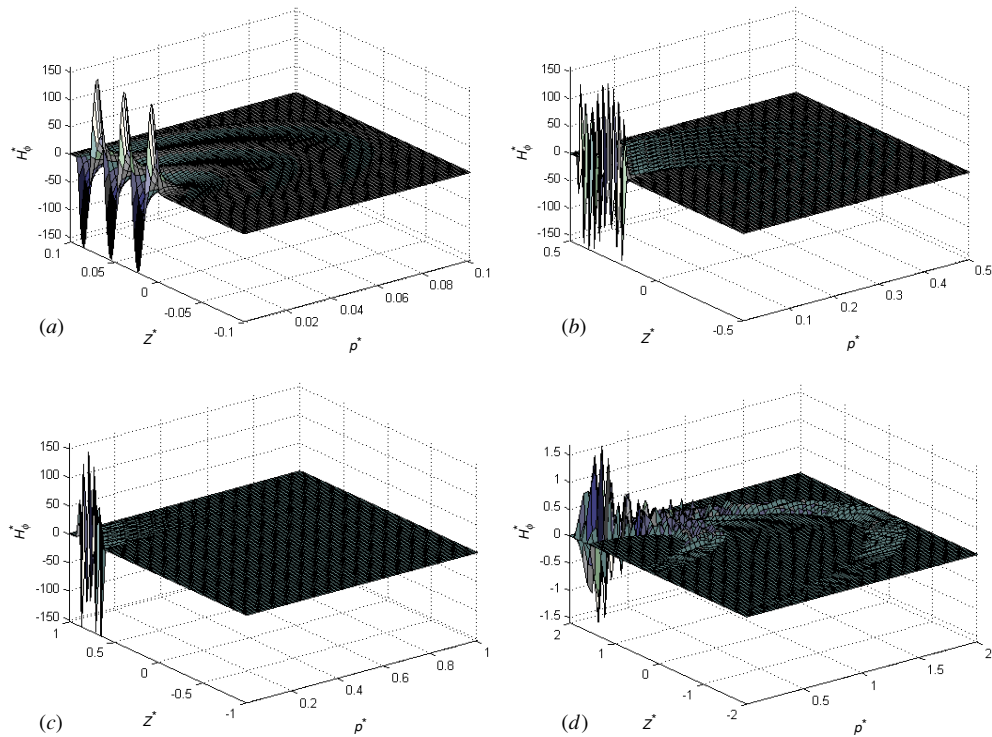
Case (iii):

$$F_1^{(iii)} = r_l + z_l \qquad \text{and} \qquad F_2^{(iii)} = F_2^{(i)}. \tag{25}$$

Case (iv):

$$F_1^{(iv)} = F_1^{(iii)} \qquad \text{and} \qquad F_2^{(iv)} = F_2^{(ii)}. \tag{26}$$

Figure 1. Emanated wave structure for the case of a rectangular envelope, $U(z, \tau) = U_0 = $ const, represented in the dimensionless form: $z^* = z/l$, $\rho^* = \rho/l$, $H_\varphi^* = H_\varphi l/U_0$. Modulation factor is $\cos[k(\tau - z)]$, $kl = 200$, $T/l = 0.3$, $\beta = 0.9$; $\tau/l = 0.1$ (a), 0.5 (b), 1 (c) and 2 (d).

Assuming that $q(x) = U(x)/x$ and $q'(x)$ are the continuous and absolutely integrable functions, respectively, we obtain from expression (22) the relation

$$\Psi_+ \cong \frac{i}{4\pi ck} \exp(ik(\tau + z)) \left\{ \frac{1}{F_1} U \left(\tau - z - \frac{\rho^2}{F_1}, \tau + z - F_1 \right) e^{-ikF_1} \right.$$
$$\left. - \frac{1}{F_2} U \left(\tau - z - \frac{\rho^2}{F_2}, \tau + z - F_2 \right) e^{-ikF_2} \right\} + o \left(\frac{1}{kl} \right). \tag{27}$$

Hence, using relation (3), one gets the approximate expression for the magnetic strength component

$$H_\varphi \cong \frac{i}{4\pi} \exp(ik(\tau + z)) \left\{ \frac{1}{F_1} \frac{\partial F_1}{\partial \rho} U \left(\tau - z - \frac{\rho^2}{F_1}, \tau + z - F_1 \right) e^{-ikF_1} \right.$$
$$\left. - \frac{1}{F_2} \frac{\partial F_2}{\partial \rho} U \left(\tau - z - \frac{\rho^2}{F_2}, \tau + z - F_2 \right) e^{-ikF_2} \right\} \tag{28}$$

which is the exact solution of the electrodynamic problem for the case when the amplitude $U(z, \tau)$ in the expression for the current distribution (21) is taken as the function of the variable $\tau + z$ only. Then the factors U in the above relation are the functions of one argument

$\tau + z - F_{1,2}$. Depending on the interrelations between the radiator parameters, the observation time and location, the H_φ component of the field is described by the following expressions:

Case (i):

$$H_\varphi \cong \frac{1}{4\pi} \frac{r_\beta - z_\beta}{r_\beta \rho} U(\beta) \exp\left[ik \left(\frac{1+\beta}{1-\beta}\right)^{1/2} (\tau_\beta - r_\beta) \right]$$

$$- \frac{1}{4\pi} \frac{r - z}{r\rho} U(0) \exp(ik(\tau - r)) = H_{\varphi+}(\beta) - H_{\varphi+}(0). \tag{29}$$

Case (ii):

$$H_\varphi \cong H_{\varphi+}(\beta) - \frac{1}{4\pi} \frac{r_T - z_T}{r_T \rho} U(T) \exp\left\{ ik \left[T + \left(\frac{1+\beta}{1-\beta}\right)^{1/2} (\tau_T - r_T) \right] \right\}$$

$$= H_{\varphi+}(\beta) - H_{\varphi+}(T). \tag{30}$$

Case (iii):

$$H_\varphi \cong \frac{1}{4\pi} \frac{r_l - z_l}{r_l \rho} U(l) \exp(ik(\tau - r_l + l)) - H_{\varphi+}(0) = H_{\varphi+}(l) - H_{\varphi+}(0). \tag{31}$$

Case (iv):

$$H_\varphi \cong H_{\varphi+}(l) - H_{\varphi+}(T). \tag{32}$$

The peculiarities of the wave structure generated by the current distribution I_+ for the case of a rectangular envelope (analogous to that illustrated in the previous section for the current distribution I_- in figure 1) are depicted in figure 2. General results yielded by relations (29)–(32) are the following:

1. The angular factors R_+ in the above expressions and the factors R_- of section 3 satisfy the relation $R_+ = 1/R_-$. Hence, the angular distributions of the waves generated by the sources with the total current I_- and I_+ differ essentially. For example, in case (i) the coefficients $R_+(\theta_\beta)$ and $R_+(\theta)$ are equal to zero when $\theta_\beta = \theta = 0$ (as it was mentioned earlier, in this situation $R_-(\theta_\beta)$ and $R_-(\theta)$ are infinite, see section 3).

2. In sections 3 and 4, the factors U have the identical arguments for the corresponding cases.

3. The directionality of the waves produced by the current pulse described by expression (1) involving the continuous factor (21) is observed for the angles close to π, i.e. in the direction opposite to that of the current-pulse propagation (see figure 2(*d*) illustrating the wave dynamics after the end of the current pulse, $\tau > T + l/\beta$).

4. One can see the frequency transform in the cases (i), (ii) and (iv). The limiting values of the frequency are $\omega((1+\beta)/(1-\beta))$ if $\theta = 0$ and ω if $\theta = \pi$ ($\beta \neq 1$). Here, we have the frequency shift towards the violet spectrum region (the violet shift), which is maximal in the direction of the pulse front movement. The special case of $\beta = 1$ requires particular consideration.

5. For $\beta = 1$, the spacetime structure of the waves generated by the total current I_+ requires the detailed investigation.

5. The far-field radiation characteristic and the frequency transform

Here, we discuss the essential peculiarities of the generated waves, namely, the directionality and the frequency transform. We compare the waves produced by the *long* and *short* source-current pulses in the case of both the total current distributions I_- and I_+. The special features
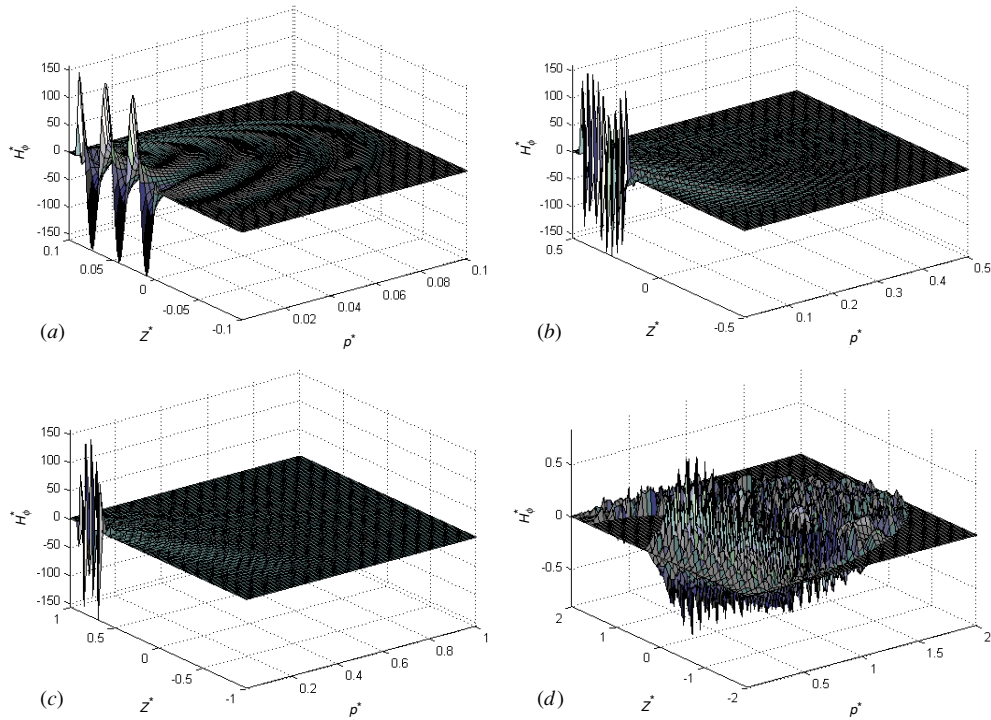
Figure 2. Emanated wave structure for the case of a rectangular envelope, $U(z, \tau) = U_0 = $ const, represented in the dimensionless form: $z^* = z/l$, $\rho^* = \rho/l$, $H_\varphi^* = H_\varphi l/U_0$. Modulation factor is $\cos[k(\tau + z)]$, $kl = 200$, $T/l = 0.3$, $\beta = 0.9$; $\tau/l = 0.1$ (*a*), 0.5 (*b*), 1 (*c*) and 2 (*d*).

of the electromagnetic field generated by the pulses of different durations manifest themselves in the cases (ii) and (iii), therefore we give the main attention to the investigation of expressions (16), (18), (30) and (31).

5.1. Long current pulse

Let us assume that $r \gg l$, then

$$r_l \cong r\left(1 - \frac{l}{r}\cos\theta\right), \qquad r_l - z_l \cong r(1 - \cos\theta)\left(1 + \frac{l}{r}\right),$$

$$r_l \pm l \cong r\left[1 \pm \frac{l}{r}(1 \mp \cos\theta)\right]. \tag{33}$$

In case (iii), using the above relations, from expressions (18) and (30) one can obtain, respectively,

$$H_{\varphi-} \cong \frac{1}{4\pi r} R_-(\theta)\, U(0)\{\exp[ik(\tau - r)] - \exp[ik(\tau - r - l(1 - \cos\theta))]\}$$

and $\tag{34}$

$$H_{\varphi+} \cong \frac{1}{4\pi r} R_+(\theta)\, U(0)\{\exp[ik(\tau - r + l(1 + \cos\theta))] - \exp[ik(\tau - r)]\}.$$

Hence, for the intensity of the electromagnetic waves we have

$$J_- = J_0 \left(\frac{U(0)}{r}\right)^2 \cot^2 \frac{\theta}{2} \sin^2\left(kl\frac{1 - \cos\theta}{2}\right) \tag{35}$$

and

$$J_+ = J_0 \left(\frac{U(0)}{r} \right)^2 \tan^2 \frac{\theta}{2} \sin^2 \left(kl \frac{1+\cos\theta}{2} \right) \tag{36}$$

where $J_0 = (8\pi^2 \varepsilon c)^{-1}$ and ε is the permittivity of the medium.

From here on, we assume that $kl \gg 1$. The angular factors in the above expressions determine the wave directionality and the maxima and minima of the interference pattern, respectively. According to formula (35), the intensity distribution J_- is maximal for small angles θ, i.e. in the direction of the current-pulse movement (but is zero for $\theta = 0$), and decreases sharply with an increase of the polar angle. The positions of maxima of the interference pattern, θ_n, are determined by the relation $(1 + \cos\theta_n)/2 = (2n + 1)\pi/2kl$, that can be rewritten as $\sin^2(\theta_n/2) = (2n + 1)(\pi/2kl) = (2n + 1)\lambda/4l$ or in the region of the maximal directionality of radiation, where θ is small, by $\theta_n^2 = (2n + 1)\lambda/l$. Here, $\lambda = 2\pi/k$ is the wavelength of the radiation emitted by the oscillators forming the current and n is an integer.

Considering the angular dependence of the distribution J_+, let us pass to the angle $\alpha = \pi - \theta$. Then, the angular coefficient in formula (36) takes the same form as in (35), $\cot^2(\alpha/2)\sin^2((1/2)kl(1-\cos\alpha))$. Hence, we find that the maximum radiation is formed in the direction opposite to that of the movement of the current-pulse front (but $J_+ = 0$, if $\alpha = \pi$), and maxima of the interference pattern are determined by the relation $\sin^2(\alpha_n/2) = (2n + 1)\lambda/4l$ or, in the vicinity of $\theta = \pi$, by $\alpha_n^2 = (2n + 1)\lambda/l$.

The distinguished peculiarities take place for slowly varying amplitude factors of the current $U(z, \tau)$ and under the assumptions made in sections 3 and 4. Note that these peculiarities are caused by the high-frequency oscillations of the current and akin effects known for sources of the travelling-wave type. The possibility of the formation of the directional waves for $k = \infty$ ($\omega = 0$) requires separate consideration.

5.2. Short current pulse

The peculiarities of the electromagnetic field generated by the *short* current pulse, which differ from known cases, manifest themselves in case (ii). We consider the interference of the excited waves, assuming that $r_\beta \gg \beta T / \sqrt{1 - \beta^2}$. Then using the variables of the frame of reference moving with the velocity $v = \beta c$, one can obtain the approximate expressions

$$r_T \cong r_\beta + \frac{\beta \cos\theta_\beta}{\sqrt{1 - \beta^2}} T + \frac{(\beta T)^2}{2r_\beta(1 - \beta^2)},$$

$$\tau_T - r_T \cong \tau_\beta - r_\beta - \frac{1 + \beta\cos\theta_\beta}{\sqrt{1 - \beta^2}} T - \frac{(\beta T)^2}{2r_\beta(1 - \beta^2)}$$

$$r_T - z_T \cong (r_\beta - \tau_\beta)(1 - \cos\theta_\beta) + \frac{(\beta T)^2}{2r_\beta(1 - \beta^2)} \tag{37}$$

$$T + \left(\frac{1-\beta}{1+\beta} \right)^{1/2} (\tau_T - r_T) \cong \left(\frac{1-\beta}{1+\beta} \right)^{1/2} \left[\tau_\beta - r_\beta + \frac{\beta(1 - \cos\theta_\beta)}{1+\beta} T - \frac{(\beta T)^2}{2r_\beta(1 - \beta^2)} \right]$$

$$T + \left(\frac{1+\beta}{1-\beta} \right)^{1/2} (\tau_T - r_T) \cong \left(\frac{1+\beta}{1-\beta} \right)^{1/2} \left[\tau_\beta - r_\beta - \frac{\beta(1 + \cos\theta_\beta)}{1-\beta} T - \frac{(\beta T)^2}{2r_\beta(1 - \beta^2)} \right].$$

Assuming that $\beta \neq 1$, we may omit the last item in each of these relations. Being expressed via the variables τ_β and r_β, the conditions necessary for the realization of the case (ii) take the

form

$$\rho^2 + (z_\beta + \beta T/\sqrt{1-\beta^2})^2 < (\tau_\beta - T/\sqrt{1-\beta^2})^2 \qquad \text{and} \qquad (\tau_\beta - l\sqrt{1-\beta^2}/\beta)^2 < r_\beta^2 \tag{38}$$

and when $r_\beta \gg \beta T/\sqrt{1-\beta^2}$ we arrive at the approximate relation for the first inequality

$$\tau_\beta - r_\beta \geqslant \frac{T}{\sqrt{1-\beta^2}} + \frac{(\beta T)^2}{2r_\beta(1-\beta^2)}.$$

Then using relations (16) and (30), one can obtain

$$H_{\varphi-} \cong \frac{1}{4\pi r_\beta} R_-(\beta) U(\beta) \exp\left[ik\left(\frac{1-\beta}{1+\beta}\right)^{1/2}(\tau_\beta - r_\beta)\right]\left\{\exp\left[ikT\frac{\beta(1-\cos\theta_\beta)}{1+\beta}\right] - 1\right\} \tag{39}$$

$$H_{\varphi+} \cong \frac{1}{4\pi r_\beta} R_+(\beta) U(\beta) \exp\left[ik\left(\frac{1+\beta}{1-\beta}\right)^{1/2}(\tau_\beta - r_\beta)\right]\left\{1 - \exp\left[-ikT\frac{\beta(1+\cos\theta_\beta)}{1-\beta}\right]\right\}. \tag{40}$$

So, for the intensity of the electromagnetic waves we have the following expressions:

$$J_- = J_0\frac{1}{r_\beta^2}U^2(\beta)\cot^2\frac{\theta_\beta}{2}\sin^2\left(kT\frac{\beta(1-\cos\theta_\beta)}{2(1+\beta)}\right) \tag{41}$$

and

$$J_+ = J_0\frac{1}{r_\beta^2}U^2(\beta)\tan^2\frac{\theta_\beta}{2}\sin^2\left(kT\frac{\beta(1+\cos\theta_\beta)}{2(1-\beta)}\right). \tag{42}$$

The angular factors describe the directionality of radiation, which for J_- coincides with the direction of the current-pulse movement ($\theta_\beta \cong 0$) and is opposite for J_+ ($\theta_\beta \cong \pi$). The second factors describe the interference pattern. For the case of J_-, the positions of maxima of the interference pattern in the vicinity of maxima of the directionality factor (small angles θ_β) correspond to $\theta_{\beta n}^2 = (2n+1)((1+\beta)/\beta)(\lambda/T)$. As in the case of long current pulse, the wave directionality is observed in the vicinity of $\theta_\beta = \pi$ or, equally, $\alpha_\beta = 0$ ($\alpha_\beta = \pi - \theta_\beta$). Here, the maxima of the interference pattern are given by the formula $\alpha_{\beta n}^2 = (2n+1)((1-\beta)/\beta)(\lambda/T)$.

When the current-pulse velocity is small ($\beta \ll 1$), the simple relations

$$J_- \cong J_0\frac{1}{r^2}U^2(0)\cot^2\frac{\theta}{2}\sin^2\left(kT\beta\frac{1-\cos\theta}{2}\right)$$
$$J_+ \cong J_0\frac{1}{r^2}U^2(0)\tan^2\frac{\theta}{2}\sin^2\left(kT\beta\frac{1+\cos\theta}{2}\right) \tag{43}$$

are valid. In this case, for the frame of reference at rest and small angles θ or $\alpha = \pi - \theta$, the frequency shift $\Delta_\pm$ in the direction corresponding to the maximal intensity (i.e. forward for J_- and backward for J_+ with respect to the direction of the current-pulse motion) will be described, respectively, by the following relations:

$$\Delta_- \cong -\tfrac{1}{2}\omega\beta\theta^2 \qquad \text{the red shift,}$$
$$\Delta_+ \cong \tfrac{1}{2}\omega\beta\theta^2 \qquad \text{the violet shift.} \tag{44}$$

Note that for the certain relations between the parameters l, β and T, the case (ii) is not realized for polar angles in the vicinity of zero, $0 \leqslant \theta \leqslant \tilde{\theta}$, $\tilde{\theta} = \tilde{\theta}(l, \beta, T)$. Then the maximum of the radiation intensity J_- occurs for some angle $\theta_{\beta n}$ exceeding $\tilde{\theta}$ and only the higher orders

of the interference pattern can be observed, which are characterized by a smaller intensity, but a larger value of the frequency shift. This effect cannot take place, in principle, for electromagnetic waves produced by the source current I_+: here, the location of maximum of the radiation intensity always corresponds to $\theta \sim \pi$.

5.3. Beatings

From the general formulae of sections 3 and 4, one can see that for the cases (i) and (iv), in the spacetime domain determined by inequalities (7), the waves with two different frequencies are excited. This can lead to the formation of beatings. Here, we will restrict ourselves to the discussion of the case (i) for the total current I_-, since the analysis of the other cases is analogous and does not lead to the radically different results. Assuming that $\beta \ll 1$, we can obtain from relation (13)

$$H_\varphi \cong \frac{i}{2\pi r} \frac{\sin\theta}{1 - \cos\theta} U(0) \exp\left[i\left(k - \frac{\Delta k}{2}\right)(\tau - r)\right] \sin\left[\frac{\Delta k}{2}(\tau - r)\right] \quad (45)$$

where $c\Delta k = \omega\beta(1 - \cos\theta)$ is the frequency shift. Assuming that the temporal scale characterizing the resolution of the time detection system T_0 satisfies the inequalities $1/k \ll T_0$ and $T_0 \ll 1/(\Delta k)$ and, consequently, the averaged last factor in formula (45) is a slowly varying function, we arrive at the expression for the intensity distribution

$$J_- \cong J_0 \frac{1}{2r^2} U^2(0) \cot^2\frac{\theta}{2} \sin^2\left[\frac{\Delta k}{2}(\tau - r)\right] \quad (46)$$

describing the oscillations with the frequency equal to one-half of the frequency shift for the excited waves.

Note that for high velocities of the current-pulse motion, when $\beta \cong 1$, the structure of the beatings is more complicated by virtue of the substantial difference between the factors $R_-(\theta)/r$ and $R_-(\theta_\beta)/r_\beta$. In this case, in the limit $\beta \to 1$, according to relation (13), the term $H_{\varphi-}(\beta)$ depending on the velocity of the current-pulse motion tends to zero as $\beta \to 1$.

5.4. Possibility of localized wave generation

Let us discuss the expressions describing the waves generated by the current I_- in the limit $\beta \to 1$. We assume that the envelope of the current oscillations is a function of the variable $\tau - z$ only:

$$U(\tau, z) = \tilde{U}(\tau - z).$$

Remembering that the factors $R_-(\theta_\beta)/r_\beta$ and $R_-(\theta_T)/r_T$ tend to zero as $\beta \to 1$, one can see that the terms of the magnetic field component $H_{\varphi-}(\beta)$ and $H_{\varphi-}(T)$ are equal to zero. Then from the expressions (13), (16), (18) and (20) we find

Case (i):

$$H_\varphi = H_{\varphi-}(0)$$

Case (ii):

$$H_\varphi = 0$$

Case (iii):

$$H_\varphi = H_{\varphi-}(0) - H_{\varphi-}(l)$$

Case (iv):

$$H_\varphi = -H_{\varphi-}(l). \quad (47)$$

Here, the factors defining the field components are expressed via one-argument function $\tilde{U}$ as follows:

$$U(0) = \tilde{U}(\tau - z), \qquad U(l) = \tilde{U}(\tau - r_l - l).$$

If the electromagnetic waves are formed by the total current I_+ and the envelope factor of the current is the function of the variable $\tau + z$ only, $U(\tau, z) = \tilde{U}(\tau + z)$, we have essentially different result. When $\beta \to 1$, one can obtain that the factors $R_+(\theta_\beta)/r_\beta$ and $R_+(\theta_T)/r_T$ tend to $2/\rho$, whereas the functions $[(1+\beta)/(1-\beta)]^{1/2}(\tau_\beta - r_\beta)$ and $[(1+\beta)/(1-\beta)]^{1/2}(\tau_T - r_T)$ tend to the values $s = \tau + z - \rho^2/(\tau - z)$ and $s(T) = \tau - T + z - \rho^2/(\tau - T - z)$, respectively. Hence, the terms $H_{\varphi+}(\beta)$ and $H_{\varphi+}(T)$ become

$$H_{\varphi+}(\beta) = \frac{1}{2\pi\rho}\tilde{U}(s)\exp(\mathrm{i}ks) \qquad \text{and} \qquad H_{\varphi+}(T) = \frac{1}{2\pi\rho}\tilde{U}(s(T))\exp(\mathrm{i}ks(T)) \qquad (48)$$

when β tends to unity. Then from relations (29)–(32) we have

Case (i): when the spacetime domain is determined by the inequalities $0 < \tau - r < T$ and $\tau - r_l < l$

$$H_\varphi = \frac{1}{2\pi\rho}\tilde{U}(s)\exp(\mathrm{i}ks) - H_{\varphi+}(0) \qquad (49)$$

where the last term reduces to

$$H_{\varphi+}(0) = \frac{1}{4\pi}\frac{r-z}{r\rho}\tilde{U}(\tau - r)\exp(\mathrm{i}k(\tau - r)).$$

Case (ii): if $\tau - r > T$ and $\tau - r_l < l$

$$H_\varphi = \frac{1}{2\pi\rho}[\tilde{U}(s)\exp(\mathrm{i}ks) - \tilde{U}(s(T))\exp(\mathrm{i}ks(T))]. \qquad (50)$$

Case (iii): in the case of the opposite inequalities $\tau - r < T$ and $\tau - r_l > l$

$$H_\varphi = H_{\varphi+}(l) - H_{\varphi+}(0) \qquad (51)$$

where the first term reduces to

$$H_{\varphi+}(l) = \frac{1}{4\pi}\frac{r_l - z_l}{r_l\rho}\tilde{U}(\tau - r_l + l)\exp(\mathrm{i}k(\tau - r_l + l)).$$

Case (iv): finally, for $r < \tau - T < l + r_l$ and $\tau - r_l > l$

$$H_\varphi = H_{\varphi+}(l) - \frac{1}{2\pi\rho}\tilde{U}(s(T))\exp(\mathrm{i}ks(T)). \qquad (52)$$

When the spacetime domains are determined by the inequalities $r > \tau$ or $\tau - r > T + l$, the components of the electromagnetic field are equal to zero.

One can see from the above inequalities that every pulse should be classified as a *long* one when ρ tends to zero and $z > l$ [11]. Expressions (49), (50) and (51) involve the terms (48) containing the functions of the arguments s and $s(T)$, which is typical for representation of the localized waves produced by a current pulses moving along a straight line [11, 16].

6. Conclusion

Consideration carried out in this work deals with the arbitrary-shaped envelope of the source-current pulse, yielding general quadrature formulae and some simpler particular approximations for the magnetic field strength component H_φ. It is based on the method of incomplete separation of variables, which requires more elaborate treatment than the conventional approach of Rothwell *et al* [2] using the retarded potentials. However, here

the incomplete separation of variables provides a formal scheme of definitions of all possible regimes (cases) that appears in calculation of the solution. The scheme is based on *two-dimensional* analysis, involving one temporal and one spatial variable that is left non-separated. For the retarded potential, as far as the space variables are not separated, the same consideration (linked with the explicit representation of the retarded arguments) should be done, in general, in the space of four dimensions. In the case in question, the cylindrical symmetry admits consideration of the retarded arguments in the 3D space ρ, z, τ, which, as well as the Liénard–Wiechert potential approach, leads to simple, analysable relations only for the source pulse depending entirely on the retarded argument, $I(z, \tau) = I(z - \beta\tau)$. Notably, in the case in question, the finite duration of the current pulse ($0 \leqslant \tau \leqslant T$) and the finite length of the radiative segment ($0 \leqslant z \leqslant l$) manifest themselves, explicitly or implicitly, in factors such as $h(\tau)$, $h(T - \tau)$, $h(z)$ and $h(l - z)$, so tracking of the wavefronts via the explicit representation of the integration limits in corresponding time convolutions leads to relations of the same complexity as those obtained within the scope of incomplete separation of variables.

This work is one more step towards adapting the general solution to the resulting initial-value problem, reported in [11, 17], for more and more specific and practical situations [3, 18–20], in which the solving scheme yields increasingly complicated quadrature formulae that yet define explicitly the wavefront location and are more suitable for analysis than higher order integrals or series resulted from complete separation of variables or particular numerical results. In contrast to the general consideration [11], based on representation of the current distribution $I(z, \tau)$ in the form

$$I(z, \tau) = J(z, \tau)h(z_{\text{f}}(\tau) - z)h(z - z_{\text{b}}(\tau)) \tag{53}$$

where $z_{\text{f}}(\tau)$ and $z_{\text{b}}(\tau)$ denote the coordinates of the front and back, explicit representation of the current as a pulse of duration T propagating at constant velocity βc, $z_{\text{f}}(\tau) = \beta\tau$ and $z_{\text{b}}(\tau) = \beta(\tau - T)$, and separation of the modulating factor from the envelope $U(z, \tau)$, $J(z, \tau) = U(z, \tau) \exp[ik(\tau \pm z)]$, enable one to illustrate and analytically describe the following phenomena:

- Directionality of the emanated waves.
- Transformation of the frequency of the electromagnetic wave carrier with respect to the initial frequency of the source-current modulation, which takes place for certain wave regimes and manifests itself as the red or ultraviolet shift for the modulation factors $\exp[ik(\tau - z)]$ and $\exp[ik(\tau + z)]$ correspondingly.
- In the spacetime domains determined by cases (i) and (iv) of inequalities (7), the waves of two different frequencies, fundamental and shifted, are excited, which in definite circumstances leads to the formation of beating-type interferential patterns. The structure of these beatings become more and more complicated as the current-pulse velocity tends to the velocity of light, $\beta \rightarrow 1$.

Investigation of the electromagnetic field structure was carried out only for the magnetic field strength component as definition of H_{φ} does not require knowledge of the electric charge distribution and can be made within the same scheme for all pertinent physical models of the modulated source current discussed in the introduction. Calculation of the electric induction in the near zone requires separate consideration carried out on the basis of the Whittaker–Bromwich potential rather than its temporal derivative, and yields different results for the wire antenna model and models connected with macroscopic current formation accompanying absorption of hard radiation by a medium—due to the presence of the point charges at the end points of the wire segment in the former case. Even in the simplest case, this consideration requires analytical calculations akin to those made in sections 2–4; for this

reason, consideration of the electric filed component in the near zone goes beyond the scope of the present work. In the far zone, $r \gg l$, the amplitude of the electric induction is proportional to the amplitude of the magnetic field strength while the electromagnetic energy density is proportional to H_φ^2 and thus can easily be estimated. Hence, the above discussion of the peculiarities of the emanated electromagnetic wave structure made on the basis of the magnetic component only is complete for the observation points situated sufficiently far from the radiative segment $\rho = 0, 0 \leqslant z \leqslant l$. As far as equation (6) represents the scalar wave equation in the cylindrical coordinates, all the results obtained can easily be extended to the case of scalar (e.g. acoustic) waves, the wave process being completely described by the function Ψ in both near and far zones.

It should be noted that

1. For the case of the source pulse moving at the velocity of light, $\beta = 1$, the explicit analytical solution can be obtained [17].
2. This solution depends continuously on β and can be treated as a sufficiently good approximation for the case $\beta < 1$ and $1 - \beta \ll 1$ [17].
3. For the case of different types of the sinusoidal envelope, the explicit analytical solution can be obtained by representation of the sine (cosine) factor via exponential terms, which finally leads only to some additional frequency shift in the resulting formulae. However, the final relations are too complex to be discussed in detail within the scope of this work.
4. The treatment was carried out for the case in which the dimensionless velocity of the pulse front, v/c, lies within the segment $(0, 1]$ and the oscillation wave number is $k = \omega/c$. Investigation of the general solution in the case of arbitrary phase velocity $\upsilon \in (0, \infty), k = \omega/\upsilon$ can be done within the framework of the same solving scheme. However, it requires separate consideration (that will be reported elsewhere) as the resulted waves have different structure.

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